$u = 50$, a questão coloca-se em termos de considerar o critério minimização da probabilidade de ruína ou maximização do lucro esperado retido.

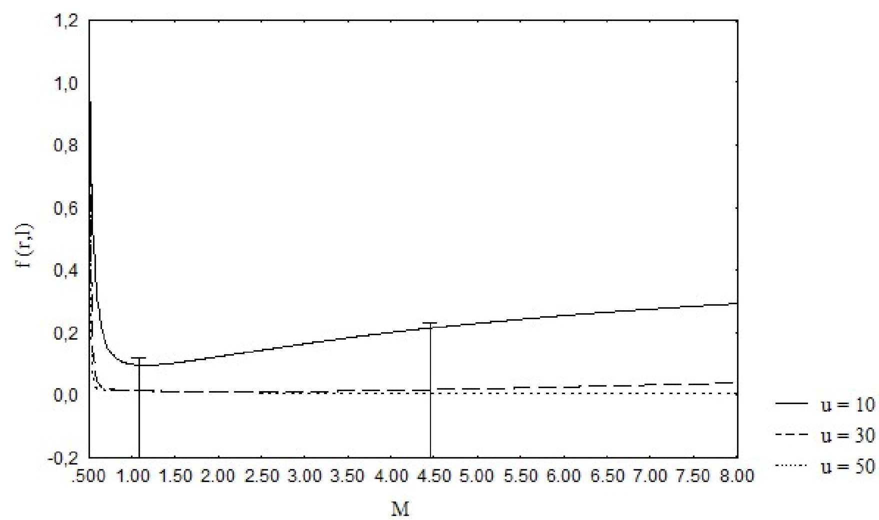

Figura 3-12 $b = 0.1$.

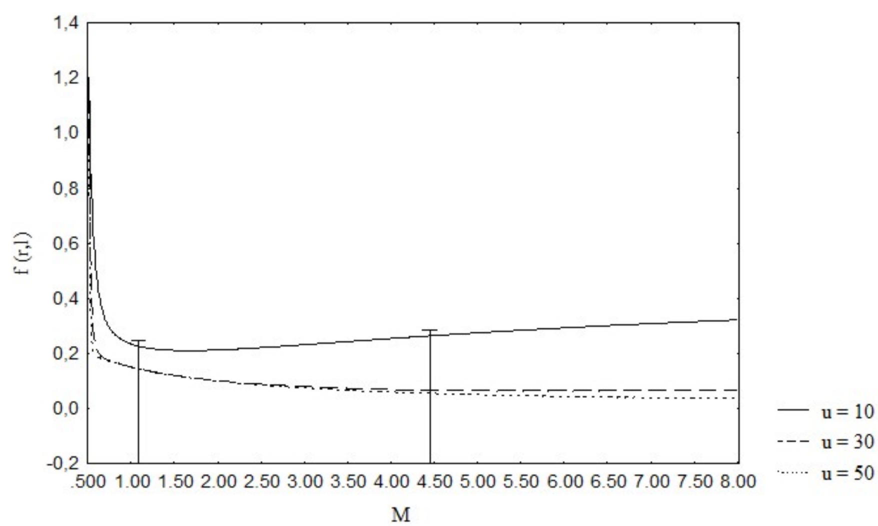

Figura 3-13 $b = 1$.

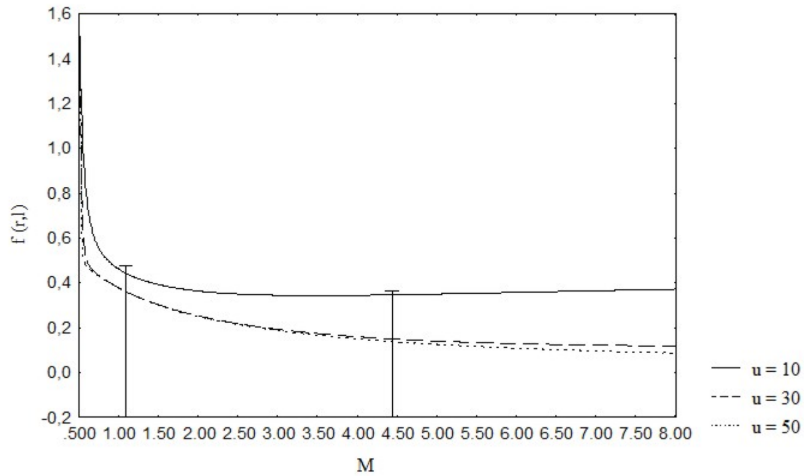

Figura 3-14 $b = 2.5.$

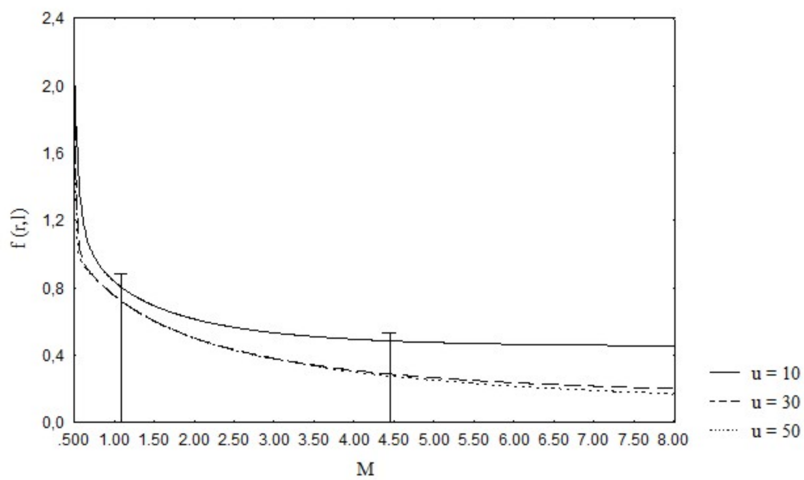

Figura 3-15 $b = 5.$

3.5.1 Conclusões

Analisando os resultados obtidos para os dois casos, observa-se graficamente que, a expressão (21) é mais influenciada pela probabilidade de ruína quando os valores de b são baixos e mais influenciada pela perda de lucro por ter feito resseguro quando o b é próximo do limite superior, como seria de esperar.

Observa-se também que, de um modo geral, no caso da reserva inicial ser 50 o b escolhido tem pouca influência, pois o limite de retenção óptimo atendendo a este critério é o $M^{\#3}$ ou muito próximo deste.

Os resultados obtidos por este critério dependem do decisor em duas situações; na escolha do $M^{\#3}$, como se viu na secção 3.4 e na escolha de b, isto é, do peso que atribui à perda de lucro com o resseguro, atendendo aos acréscimos de probabilidade de ruína versus acréscimo de lucro esperado retido.

Comparando a combinação C2 e a C4, quando se considera a combinação C2 a probabilidade de ruína, não varia tanto com b e u quanto na combinação C4, mas como se pode observar os valores da probabilidade de ruína atendendo a esta última combinação são mais baixas, decrescendo ainda mais com u.

4 Conclusão

Pretendeu-se encontrar uma solução intermédia para os decisores menos avessos ao risco, que queiram atender também à expectativa de lucro para além da probabilidade de ruína, na determinação do valor óptimo do limite de retenção, M.

Conclui-se que, a opção entre os critérios de determinação do limite de retenção óptimo está relacionada essencialmente com o nível de reserva inicial da seguradora, isto é, para uma reserva inicial baixa a utilização dos critérios clássicos é a opção mais correcta, quando a reserva inicial é adequada observa-se que o critério minimização de $\Psi^{re}(u) + b[E[W] - E[W_{ret}]]$ apresenta bons resultados na escolha do limite de retenção óptimo, podendo o decisor optar entre os critérios clássicos (minimização da probabilidade de ruína ou na sua versão mais prática a maximização do coeficiente de ajustamento) e este.

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The Immunisation of a Workers' Compensation Fund

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ABSTRACT

The aim of this study is to present a comparison of the results of the immunisation of a Workers' Compensation Pension Fund in Portugal obtained by using a deterministic model with those obtained by using a stochastic model developed by Longstaff and Schwartz (1993,1992).

We observe that the use of a mandatory interest rate of 6% and the French Mortality Table TV 73-77 to calculate the liabilities, as obliged by law, undervalues the real amount of the pension flows. This is because the upward and downward movements of the interest rate are parallel and affect simultaneously all the terms of the structure of the interest rate.

Therefore, we recalculated these obligations using the Longstaff and Schwartz model, in which the variations of the interest rate are stochastic, and we conclude that the amount of the liabilities is higher than that resulting from the deterministic model, due to the volatility of the short-term interest rate.

Keywords: Workers' Compensation Pensions; Immunisation; Stochastic duration; Affine term-structure models.

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1 Introduction

In Portugal, Workers' Compensation insurance is a legal obligation on the part of employers (included in the Social Security benefits). Employers are obliged to transfer liability as defined by law to bodies authorized and specialized in the management of this form of insurance (i.e. general insurance companies - third-party liability insurance), should the former lack sufficient funds to assume, on their own account, the cost of temporary or permanent physical disability arising from accidents in the work-place. Likewise, as of 1 November 1999, the self-employed are obliged to abide by the same legal requirements.

The insurance guarantees that those insured are covered for accidents that occur as a consequence of their carrying out their occupational activity.

The purpose of the coverage is to assure the conditions of survival and recovery of the victims of an accident, as well as the welfare of their relatives.

As a consequence of the accident, the worker can die, be temporarily disabled or acquire a permanent disability. In the first case, his/her relatives will be entitled to receive either a lump sum or a life annuity. In the second case, the injured person will receive an allowance for temporary incapacity to work and also medical, surgical, pharmaceutical or other care, if needed. In the latter case, the injured will also receive a pension. Workers' Compensation contracts are drawn up company by company, regardless of the number of employees to be covered, and premiums are negotiated case by case.

A Workers' Compensation fund aims to provide compensation for disabilities suffered, by means of benefit payments as defined by law, which, in the case of permanent incapacity, can only be attributed in the form of a pension for life[1], decided by the Labour Tribunal and which, on reaching the age of retirement, continues to be paid in addition to the State Social Security pension.

Should the accident result either in a permanent reduction of the injured party's work-or earning-capacities or, indeed, the death of the victim, then Decree-Law number 100/97, 13.9, stipulates an entitlement to specified benefits (i.e. pensions) to the victim or his/her surviving marital partner. This benefit is annulled in the event of remarriage.

[1] In fact, it is an annuity calculated as in life assurance, taking into account the age of the beneficiary and a mortality table.

These pensions can, under certain, stipulated circumstances, also be attributed to surviving, dependent parents and/or children of the deceased.

The effects of the variations of the interest rate on the duration of the liabilities of a general insurance company, were analysed by Babbel and Klock [1994][2], identifying and quantifying the cash-flows of the liabilities based on the application of the development factors of the reserves. They propose the calculation of modified duration of the liabilities, based on the definition of transformed cash-flows. This methodology was also applied to the Workers' Compensation business, taking into consideration the long-term nature of the insurance company's commitments, namely, in the pensions domain, in which the duration of the liability is, obviously, unspecifiable.

Therefore, it is crucial to analyse the conditions for the immunisation of a mutual fund for Workers' Compensation in Portugal, based on the deterministic principles proposed by Redington [1952][3] and revised by Boyle [1976][4] for the hypothesis of stochastic movements of the interest rates in accordance with the modelisation by Cox, Ingersoll and Ross [1979][5].

These results will subsequently be compared with the model by Longstaff and Schwarz [1992], which considers two factors in the structure of the interest rate, namely, the inherent dynamics of the short-term interest rate and its volatility.

The present work is organised as follows:

In section 2, we analyse the effects of the changes either in the mortality rate or in the interest rate, which, in turn, determine changes to the cash-flows and to the measures of duration and convexity in a deterministic model. In section 3, the deterministic model of immunisation of a Workers' Compensation fund is proposed, with the inclusion of an empirical application. This model is compared with the stochastic models of Cox, Ingersoll and Ross and of Longstaff and Schwarz in section 4.

The data and the methodology are explained in section 5 and an empirical

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application of the stochastic model to a real fund is presented in section 6. The main conclusions are given in section 7.

2 Fund Management

In this section, we briefly describe the historical results of the Workers' Compensation line of business.

The funds corresponding to the permanent disability pensions are managed by insurance companies authorised to operate general lines of business in Portugal.

In 1997, the mathematical provisions corresponding to the underwritten liabilities (pensions, presumed pensions and pension reinforcement) totalled 141.336 million contos[6] (705 million euros).

Approximately 40 million contos (200 million euros) of provisions were added to this figure to cover the evolution of Workers' Compensation claims and unearned premiums, raising the total of funds invested to approximately 181.333 million contos (904 million euros) in a portfolio of financial assets composed primarily of Government and corporate bonds.

2.1 The Cash-Flows of a Workers' Compensation Fund

In general terms, we can say that the periodical cash-flow of a Workers' Compensation fund managed by an insurance company, calculated by the subtractive method, is based on the following:

[6] (n.b. 1 conto=1000 PTE).

+	*Premiums*
-	*Cash payments* (reimbursement of expenses)
-	*Payments of indemnities (temporary incapacity)* Replacing the salary.
-	*Pension payments (permanent incapacity, death)*
-	*Variation of the technical reserve for outstanding claims*
-	*Operating costs of the company*
+	*Financial income* Interest, dividends, gains from transactions on the portfolio of assets.
-	*Reinsurance*
=	*Final result*

The earned premiums are equal to the written premiums, less reimbursements and cancellations deducted from the unearned premiums reserve (premiums to cover risks in the following year).

The technical reserve for outstanding claims is defined as the predicted total of future costs of all claims having been incurred up to the date of the balance sheet. Moreover, those claims which have been incurred, but not yet reported (i.e. IBNR) by the date of closure of the balance sheet must also be taken into account.

These reserves for this particular line of business can be divided in two main categories:

1. Reserve for indemnities not related to pension payments (e.g. medical expenses, salaries indemnities for partial disability, etc). This reserve was calculated traditionally in the Portuguese market as a percentage of at least 25% of the commercial premiums and handling charges in the financial year, less reimbursements and cancellations;

2. The mathematical reserves corresponding to the pensions due to permanent disabilities. These reserves are equal to the present value of the pensions to be paid out owing to Workers' Compensation claims, calculated in accordance with the regulations in force.

In calculating the outstanding claim reserves provision, the expenses incurred in handling the claims are also to be taken into account, irrespective of their origin.

The problem, therefore, is to transform the technical reserves, included in the balance sheet as a single amount, into periodical flows.

With regard to the reserves for indemnities other than pensions (i.e. operating costs, payments of allowances, medical and hospital expenses and others) and the pension payments, this transformation is simple.

In contrast, as far as the mathematical reserves for pensions are concerned, this transformation will obey a specific actuarial methodology.

2.2 The Market Operating Account for Workers' Compensation Insurance

The operation of this line of business presented the following evolution during the period analysed:

TABLE 1

Workers' Compensation Line				
Market			U: millions PTE	
	1990	**1995**	**1996**	**1997**
Written Premiums	**50.718**	**75.327**	**78.432**	**79.31**
Earned Premiums		74.626	77.692	79.068
Loading	40.724	58.808	63.774	66.244
Underwriting Margin	9.994	15.818	13.918	12.824
Operating Costs	20.378	21.991	23.852	22.626
Net Margin	-10.384	-6.173	-9.934	-9.802
Reinsurance balance	**107**	**-136**	**-42**	**-88**
Operating Result	**-10.277**	**-6.31**	**-9.976**	**-9.89**
Financial Income	**11.952**	**12.801**	**14.947**	**21.649**
Final Result	1.675	6.491	4.971	11.759
Number of Claims	n.a.	94.49	113.569	231.092

Sources:
1990 - ISP, *"Actividade Seguradora em Portugal 1990"*
1995 - ISP, *"Estatísticas de Seguros 1995"*
1996 - ISP, *"Estatísticas de Seguros 1996"*
1997 - APS, *"Relatório de Mercado 1997"*

As we can see in Table 1, the net margin of the line is always negative, but the final result is always positive. This occurs because of the financial gains that cover completely the operating result.

Taking the premiums as a base 100, we can confirm that the underwriting margin never covers the operating costs in all the period analyzed, owing fundamentally to the technical costs and within these, to the costs of the provision for future pensions. However, the net margin remains stable over the last two years because of the decrease in the rate of the operating ratio. We must stress that due to the consequences of the management of the portfolio of financial assets, the final result more than quadruples the result of 1990.

TABLE 2

Workers' Compensation Line				
Market				
	1990	**1995**	**1996**	**1997**
Written Premiums	100,0%	100,0%	100,0%	100,0%
Earned Premiums		99,1%	99,1%	99,7%
Loading	80,3%	78,1%	81,3%	83,5%
Underwriting Margin	19,7%	21,0%	17,7%	16,2%
Operating Costs	40,2%	29,2%	30,4%	28,5%
Net Margin	-20,5%	-8,2%	-12,7%	-12,4%
Reinsurance Balance	0,2%	-0,2%	-0,1%	-0,1%
Operating Result	**-20,3%**	**-8,4%**	**-12,7%**	**-12,5%**
Financial Income	23,6%	17,0%	19,1%	27,3%
Final Result	3,3%	8,6%	6,3%	14,8%

This result is all the more striking, given that the line of business is a social, mandatory insurance.

We stress that the beneficiaries have no participation in the financial results, other than the mandatory interest rate (6% until the end of the Financial Year 1999).

3 The Model of Deterministic Immunisation of Workers' Compensation Pensions

One of the main purposes of this section is to assess the effect of the variations of mortality and/or the interest rate on a Workers' Compensation pensions portfolio.

This line of business is characterised by the existence of a time-lapse between the payment of the premium by the policyholder and the dates of payments for claims, in particular those claims which will give rise to the life annuities which will continue to be paid even when the pensioner becomes eligible for a State Social Security pension.

The existence of this time-lapse permits the insurance company to obtain returns from the management of the portfolio of covering assets from the mathematical reserves. Since the insurance contracts involve cash-flows at different points of time, the models traditionally used in finance are a logical point of departure for an analysis of the immunisation problem.

To cover the liabilities derived from the life annuities, the insurance company is obliged to include in its balance sheet an amount equivalent to a single premium corresponding to a life annuity, calculated at a technical interest rate of 6% and in accordance with the official mortality rate table.[7]

In this way, Workers' Compensation pensions are the equivalent of immediate annuities sold by Life Insurance companies.

This equivalence makes it possible, therefore, to respect the two conditions of Redington's immunisation. The first condition is that the duration of the assets must be equal to that of the liabilities, and the second states that the dispersion of the cash-flows of the assets should be greater than the dispersion of the cash-flows of the liabilities.

However, in order to obtain the second condition, the majority of the works published on the subject consider that this condition is verified immediately if, as proposed by Bierwag, Kaufman and Toevs [1983][8], certain rules can be empirically proven. According to the latter authors, in order to have immunisation, both of the following are required:

- the durations of the sub-sets of assets and those of the sub-sets of the

[7] French female population for the period 1960-64.

[8] Bierwag,G., Kaufman,G. and Toevs,A. (1983), Immunisation Strategies for Funding Multiple Liabilities, Journal of Financial and Quantitative Analysis, Vol.18, Nº1.

liabilities should be equal;

- the durations of the sub-sets of assets are respectively shorter than the shortest term of the sub-set of the liabilities and longer than the longest term of the sub-set of the expected liabilities.

Furthermore, taking into account the influence of the reinsurance, the correction of the cash-flows can have a decisive bearing on the figures of duration and convexity, since the variations of the cash-flow derived from reinsurance could be negatively correlated with the variation generated by fluctuations in the interest rate, thereby neutralizing the correction.

3.1 The Assets/Liabilities Immunisation

The evaluation of the liabilities arising from the payment of life annuities depends mainly on two factors:

- the interest rate, r, used in the calculation of the present value of the cash-flows;

- the probability of survival, ${}_tp_x$, implicit in the calculations made with a given mortality table.

For a particular individual, if his/her life expectancy were n, he/she would receive the series of certain annuities:

$$a_{\overline{1|}\,r}, a_{\overline{2|}\,r}, a_{\overline{3|}\,r}, \cdots a_{\overline{n|}\,r}$$

with probability:

$$P(\hat{a}_x = a_{\overline{n|}\,r}) = \left(\frac{d_{x+n}}{l_x}\right), (n \geq 0)$$

Since it is a question of a random value, Pollard et al. [1969][9] define the variable moments as the moment of order r, in relation to the origin and given by the expression:

$$E\left((\tilde{a}_x)^r\right) = \sum_{n=0}^{\infty} (a_{\overline{n|}})^r \left(\frac{d_{x+n}}{l_x}\right)$$

[9] Pollard, A., Pollard, J (1969), A Stochastic Approach to Actuarial Functions, Journal of The Institute of Actuaries, Vol. 95, pp. 79-113 (1969).

and the moment of order one is the value of a_x.

At the outset of the process, the company's expected net present value, according to Pollard et al, is equal to:

$$E\left(a_{\overline{n}|} - \sum_{k=0}^{w} a_{\overline{n}|} \frac{d_{x+n}}{l_x}\right) = a_{\overline{n}|} - a_x$$

with n as the life expectancy (e_x) of the pensioner of initial age x.

It is demonstrated that the value of the uncertain annuity a_x is always lower than the value of the certain annuity calculated with maturity, n, equal to the pensioner's life expectancy at age x.[10]

On the other hand, Jordan [1975][11] demonstrates that when there is constant change in the force of mortality, its effect on an annuity is equivalent to that of the same change on the continuous interest rate. This being so, we can treat the effect of the variation of the interest rate by acknowledging that an identical variation is observed in the mortality.

In an insurance portfolio, the above expression represents the net value of the Workers' Compensation fund, and we observe that, under the conditions described below, this value is always positive.

Let us suppose that the flows from the pensions portfolio of the insurance company are covered by the flows from a portfolio of bonds with the same maturity.

Traditionally, the conditions required to ensure the perfect match between assets and liabilities are associated with Redington's results [1952]. A brief description of these is now presented.

(1) The present value (PV) of both assets and liabilities needs to be equal, which means

$$VA = VL$$

where $VA = \int_0^\infty A_T e^{-rT} dT, \quad VL = \int_0^\infty L_T e^{-rT} dT$ and r represents the flat discount rate under continuous compounding and a deterministic interest rate setting.

So, the profit is equal to the value of the assets from which the value of the liabilities has been deducted. The profit is designated by $S(r) = A(r) - L(r)$.

[10] cf. Neil, A. (1989), Life Contingencies, The Institute of Actuaries.
[11] Jordan, C. (1975). Life Contingencies. Society of Actuaries.

By using a Taylor series expansion to determine the impact of small changes of r on the PV of assets and liabilities, Redington proves that as long as:

(2) $$\frac{\partial VA}{\partial r} = \frac{\partial VL}{\partial r} \qquad \text{(Duration)}$$

(3) and $$\frac{\partial^2 VA}{\partial r^2} > \frac{\partial^2 VL}{\partial r^2} \qquad \text{(Convexity)} \quad ,$$

the PV of the assets will remain higher than that of the liabilities. It is important to note that by using standard duration and convexity, the previous conditions only ensure a perfect match between assets and liabilities under parallel shifts of the yield curve. In this situation, only the level risk is offset (this case is often mentioned as the portfolio dedication in the literature on the subject).

Referring to Bierwag [1987][12], it is proven that when $n \geq 0$, the derivative

$$\frac{dS}{dr} = -v\left(\sum_{t=1}^{n} t \cdot CF \cdot (1+r)^{-t} - \sum_{t=1}^{\infty} t \cdot CF \cdot (1+r)^{-t}{}_t p_x\right)$$

is nil when the probability is one, by which Redington's first condition of immunisation is respected, that is:

$$DA \cdot a_{\overline{n}|} = DL \cdot a_x$$

with

DA = *Duration of assets*
DL = *Duration of liabilities*

The second condition of immunisation is also verified, since the second derivative is positive when $DA = DL\left(\frac{L}{A}\right)$ and $(-IL + IA + DL^2 - DA^2) \geq 0$:

$$\frac{d^2S}{di^2} = v^2 \cdot (DA \cdot A - DL \cdot L - IL + IA + DL^2 - DA^2) \geq 0$$

If the two conditions are satisfied, it is proven that there are arbitrage profits derived from the effective structure of the interest rate (cf. [20][13])

[12] Bierwag, G. (1987), Duration Analysis, Ballinger Publishing Company.

[13] Shiu, E. (1987), "Asset/Liability Management: From Immunisation to Option Pricing Theory in Financial Management of Life Insurance Companies". Ed. by Cummins and Lamm-Tennant.

since, using the Taylor series expansion, the following is obtained:

$$S(r+\Delta r) = S(r) + S'(r)\Delta\delta + \frac{1}{2}S''(r)(\Delta r)^2$$

As $S''(r) > 0$ thus $S(r+\Delta r) > S(r)$.

3.2 Empirical Application of the Deterministic Model

The model was applied to the Workers' Compensation pensions portfolio data of an insurance company that has a 3% share of this line of business, operating in the Portuguese market in the Financial Year 1998, embracing all the pensions which were initiated in that year, or in the preceding years.

The projection was made of the future cash-outflows (pensions) for a ten-year period for all the liabilities in force.

The corresponding portfolio of financial assets is created with the sum of the mathematical provisions, this being the stock of assets re-evaluated at market prices at the end of each year.

3.3 The Portfolio of Pensions Liabilities

The structure of the pensioners' portfolio in Workers' Compensation according to age classification was as follows, over the last three years in the operating account:

TABLE 3

Age	Total of Monthly Pensions		(U:PTE)	Number of Pensioners by age		
Group	31.12.96	31.12.97	31.12.98	31.12.96	31.12.97	31.12.98
0 – 9	5.963.507	5.789.778	5.001.927	32	30	24
10 – 19	11.096.762	13.125.704	15.807.849	96	95	111
20 – 29	22.090.004	23.465.486	29.433.143	201	201	215
30 – 39	34.742.899	45.978.308	51.515.490	269	320	344
40 – 49	42.894.331	43.748.392	53.804.631	370	361	435
50 – 59	31.275.784	39.700.702	47.208.064	348	405	412
60 – 69	32.359.698	35.196.545	39.264.599	407	412	431
70 – 79	7.356.627	8.070.257	10.469.115	227	226	246
80 – 89	1.123.112	1.745.811	1.872.598	84	86	90
90 – 96	7.597	17.316	17.565	4	9	10
Total	**188.910.321**	**216.838.299**	**254.394.981**	**2.038**	**2.145**	**2.318**

The mathematical provisions constituted to cover the liabilities of the pensions to be paid, according to the age groups, were as follows over the last three years:

TABLE 4

Age	Mathematical Provision		
Group	31.12.96	31.12.97	31.12.98
0 – 9	67.263.026	65.686.882	57.335.136
10 – 19	83.214.636	99.824.475	129.365.870
20 – 29	318.645.525	339.478.562	395.522.103
30 – 39	544.254.855	720.671.206	808.149.830
40 – 49	636.612.475	687.791.903	799.476.160
50 – 59	428.040.715	521.632.326	646.834.548
60 – 69	372.285.794	366.051.927	448.002.513
70 – 79	59.358.399	54.690.202	84.419.860
80 – 89	5.748.030	7.978.001	8.803.490
90 – 96	21.563	20.299	50.367
Total	**2.515.445.018**	**2.897.387.831**	**3.377.959.877**

Appendix I presents a table of the evolution of the mathematical provisions and the pensions to be paid to Workers' Compensation pensioners as of 31.12.98 until 2009[14], calculated on a basis of all the pensioners registered on 31.12.98 and taking into account the mortality of the pensioners in each year according to the Mortality Table TV 73/77, with a technical interest rate of 6%.

3.4 The Cash-Flows of Financial Assets and Liabilities

The portfolio cash-flow of financial assets of the insurance company to cover the liabilities of the Workers' Compensation line was as follows:

[14] This is due to the portfolio of assets we used to cover the liabilities.

TABLE 5

Years	Cash-flow of assets	Net value at interest rate of 6%	Cash-flow of liabilities	Net value at interest rate of 6%
1	253.401.230	239.057.764	289.740.687	273.340.271
2	934.552.496	831.748.395	287.640.064	255.998.633
3	177.780.870	149.268.247	281.977.871	236.754.058
4	378.584.511	299.874.392	279.660.274	221.517.131
5	169.185.170	126.425.001	276.240.783	206.423.183
6	169.185.170	119.268.869	272.995.333	192.450.938
7	569.081.738	378.471.858	269.080.233	178.953.723
8	323.501.038	202.968.553	266.077.870	166.940.547
9	553.931.038	327.870.930	261.290.751	154.657.594
10	2.491.591.420	1.391.291.635	3.068.552.293	1.713.463.573
	Total	**4.066.245.643**		**3.600.499.651**

As can be observed, the initial net value of the fund is 465.745.992 PTE.

3.5 The Assets/ Liabilities Immunisation

The durations[15] of the liability and asset cash-flows and the respective dispersions are presented in the following table:

TABLE 6

	Duration	Duration*Present Value	Dispersion
Assets	**6,04**	**26.034.401.230**	**12**
Liabilities	**6,73**	**24.229.475.746**	**10,88**

So, we can confirm that, in the case of the deterministic model, the fund is immunised because the two conditions mentioned in section 3.2, i.e. $VA \cdot DA \geq VL \cdot DL$ and $IA \geq IL$ are verified. This means that the net value due to the variations of the interest rate is always positive.

[15] The duration and the convexity of the liabilities were calculated for the total number of pensioners on 31 December 1998, taking into account the mortality of the pensioners in each year, according to the Mortality Table TV 73/77, with a technical (mandatory) interest rate of 6%.

4 The Stochastic Approach

4.1 Cox, Ingersoll and Ross (1985) and Longstaff and Schwartz (1992) models.

We will now evaluate the impact on the net present value of the insurance liabilities of the assumption that interest rates follow a stochastic process. In this sense, the first point that needs to be established relates to the choice of the stochastic models to be implemented. In the early 1980s, one of the best known models to explain the term structure of interest rates was Cox, Ingersoll and Ross [CIR (1985)]. The main assumption behind this model is that the volatility of the short rate increases with the square root of the level of the rate. Taking this formulation into account, the existence of negative interest rates is excluded from the set of the model results and more variability is allowed at times of high interest rates and less variability when rates are low. The stochastic process followed by the short-term interest rate ("r") under the CIR model is:

$$dr = \alpha(\overline{r} - r)dt + \sigma\sqrt{r}\, dz$$

which implies that the general solution for pure discount bond prices, spot yields and their volatility structure are, respectively:

$$P(r,\tau) = A(\tau)e^{-B(\tau)r},$$

$$Y(\tau) = -\frac{\ln A(\tau)}{\tau} + \frac{B(\tau)}{\tau},$$

$$\sigma_R(\tau) = \frac{\sigma\sqrt{r}}{\tau}\, B(\tau),$$

where

$$\tau = s - t,\ A(\tau) = \left(\frac{\phi_1 e^{\phi_2(\tau)}}{\phi_2\left(e^{\phi_1(\tau)} - 1\right) + \phi_1}\right)^{\phi_3},\ B(\tau) = \left(\frac{e^{\phi_1(\tau)} - 1}{\phi_2\left(e^{\phi_1(\tau)} - 1\right) + \phi_1}\right),$$

$$\phi_1 \equiv \sqrt{\alpha^2 + 2\sigma^2}, \qquad \phi_2 \equiv \frac{(\alpha + \phi_1)}{2}, \qquad \phi_3 \equiv \frac{2\alpha\overline{r}}{\sigma^2}.$$

Despite the fact that the CIR model continues to be widely used at the present time, there are strong constraints on it. The most important of these relates to the incapacity of the model to fit different types of shapes for the yield curve. Given its functional form, especially its limited number of degrees of freedom, the CIR model presents only a good fit for upward term structures of interest rates. In fact, this is the main reason why this model has been so widely used over the last decade by European core markets, but not on the peripheral markets (for further details see Rebonato [1997] Chapter 9). Since the current yield curve for Euroland rates allows its projection, we decided to use this. Nevertheless, our main aim in using the CIR model was to compare its outcomes with those which we obtained from the Longstaff and Schwartz model (L&S) [1992]. This second model can be seen as an extension of the CIR model given that it uses two state variables. All the latest theoretical developments to explain the dynamics of the yield curve suggest the existence of three important explanatory variables: level, slope and curvature risks. According to the principal component analysis (PCA), these variables will correspond directly to the first, second and third components, which, in most cases, explains roughly 99% of the yield curve volatility. On the strength of these arguments, we found good support for the implementation of a two-factor model, such as that of Longstaff and Schwartz [1992]. Furthermore, given the high number of degrees of freedom presented by L&S' closed-form solution for the price of pure discount bonds, this model fits perfectly well across different yield curve shapes.

The L&S model originates from a general equilibrium framework where the short-term rate r is defined by $r = \alpha x + \beta y$ with $\alpha \neq \beta$. Alongside this, the short rate volatility ν is given by $\nu = \alpha^2 x + \beta^2 y$. Finally, the state variables x and y follow the stochastic differential equations presented below:

$$dx = (y - dx)dt + \sqrt{x}\, dz_1$$

$$dy = (\eta - \theta y)dt + \sqrt{x}\, dz_2$$

Combining these two equations with the identity for r and ν allows the authors to determine the processes for r and ν and to reach the following closed-form solution for the price of pure discount bonds:

$$P(r, \nu, \tau) = A^{2y}(\tau) B^{2\eta}(\tau) e^{(k\tau + C(\tau)r + D(\tau)\nu)} \; ,$$

where

$$\tau = s - t$$

$$A(\tau) = \frac{2\phi}{(\delta+\phi)(e^{\phi\tau)}-1)+2\phi}$$

$$B(\tau) = \frac{2\psi}{(\theta+\psi)(e^{\psi\tau)}-1)+2\psi}$$

$$C(\tau) = \frac{\alpha\phi(e^{(\psi\tau)}-1)\ B(\tau) - \beta\psi(e^{(\phi\tau)}-1)\ A(\tau)}{\phi\psi(\beta-\alpha)}$$

$$D(\tau) = \frac{\psi(e^{(\phi\tau)}-1)\ A(\tau) - \phi(e^{(\psi\tau)}-1)\ B(\tau)}{\phi\psi(\beta-\alpha)}$$

and

$$\theta = \xi + \lambda, \quad \phi = \sqrt{2\alpha+\delta^2}, \quad \psi = \sqrt{2\beta+\theta^2}, \quad \kappa = \gamma(\delta+\phi) + \eta(\theta+\psi).$$

The yield of a bond maturing in $T, Y(T)$, is simply obtained as:

$$\begin{aligned} Y(T) &= -\frac{\log(P(T))}{T} \\ &= -\left(\frac{\kappa T + 2\gamma \log A(T) + 2\eta \log B(T) + C(T)r + D(T)v}{T}\right) \end{aligned}$$

Finally, the instantaneous volatility of a bond return is given by:

$$\begin{aligned} \sigma_R(\tau) &= \frac{1}{\tau}\left[\left(\frac{\alpha\beta\psi^2(e^{\phi\tau}-1)^2A^2(\tau) - \alpha\beta(e^{\psi\tau}-1)^2B^2(\tau)}{\phi^2\psi^2(\beta-\alpha)}\right) r + \cdots \right. \\ & \left. \cdots + \left(\frac{\beta\phi^2(e^{\psi\tau}-1)^2B^2(\tau) - \alpha\psi^2(e^{\phi\tau}-1)^2A^2(\tau)}{\phi^2\psi^2(\beta-\alpha)}\right) v\right]^{\frac{1}{2}} \end{aligned}$$

4.2 Matching of Assets and Liabilities under CIR and L&S Models

In a stochastic interest rate environment, only the first order conditions (defined in terms of "stochastic durations") are relevant for formulating immunisation rules. In order to generalize the analysis, while encompassing the previously two-term structure models as special cases, the short-term interest rate, r, is defined as an affine function of a Markovian vector $\underline{X} \in \Re^n$ of n state variables:

$$r = f + \underline{G}' \cdot \underline{X},$$

where $f \in \Re$ and $\underline{G} \in \Re$ are model parameters to be estimated, and " $\cdot$ " denotes the inner product in $\Re^n$. Let us further assume that, under the probability space $(\Omega, \Re, Q)$, the state vector satisfies the following stochastic differential equation (SDE):

$$d\underline{X} = \underline{\mu}(\underline{X})dt + \sigma(\underline{X}) \cdot d\underline{W^Q}\ ,$$

where $\underline{\mu} \in \Re^n$ and $\sigma(\underline{X}) \in \Re^{n \times n}$ satisfy the usual Lipschitz and growth conditions required for a strong solution to exist, while $\underline{W^Q} \in \Re^n$ is a standard, Q-measured Brownian motion. The martingale measure Q is obtained when the "money-market account" is taken as the numeraire of the inter-temporal stochastic economy under analysis, and it is assumed to exist.[16] From Duffie and Kan (1996), it is well understood that pure discount bond prices are only exponential-affine if, and only if, both the drift and the instantaneous variance of the above diffusion process are written as affine functions of the state vector, i.e:

$$\underline{\mu}(\underline{X}) = a \cdot \underline{X} + \underline{b}\ ,$$

and

$$\sigma(\underline{X}) = \sum \times \operatorname{diag}\{\sqrt{v_1}, \cdots, \sqrt{v_n}\}$$

where $a, \sum \in \Re^{n \times n}, \underline{b} \in \Re^n$, and

$$v_i = \alpha_i + \underline{\beta_i}' \cdot \underline{X}\ \ ,\ \ i = 1, \cdots, n\ ,$$

with $\alpha_i \in \Re$ and $\underline{\beta_i} \in \Re^n$. Note, for instance, that the Longstaff and Schwartz (1992) model can be fitted into the Duffie and Kan (1996) framework through

[16] That is, the market is assumed to be arbitrage-free.

the following restrictions: $n = 2, f = 0$, matrix "a" is diagonal, matrix $\sum$ is a (2x2) identity matrix, $\alpha_i = 0, \forall i$ and $\underline{\beta_i}$ is a vector of zeros except for its i^{th} entry which is equal to one. The CIR model can also be fitted into the Duffie and Kan (1996) specifications, since this is the most general formulation for the affine class of term structure models.

Taking into account the general SDE satisfied by the state-vector, and applying Itô's lemma to the present value of both assets (VA) and liabilities (VL), it follows that[17]

$$dVA = r \times VA \times dt + \frac{\partial VA}{\partial \underline{X}'} \cdot \sigma(\underline{X}) \cdot d\underline{W^Q}$$

and

$$dVL = r \times VL \times dt + \frac{\partial VL}{\partial \underline{X}'} \cdot \sigma(\underline{X}) \cdot d\underline{W^Q} .$$

Joining the previous two equations, we obtain the dynamics of the financial institution rate of return:

$$\frac{dVA}{VA} - \frac{dVL}{VL} = \left(\frac{\partial VA}{\partial \underline{X}'} \frac{1}{VA} - \frac{\partial VL}{\partial \underline{X}'} \frac{1}{VL} \right) \cdot \sigma(\underline{X}) \cdot d\underline{W^Q} .$$

It is then clear that, starting from perfect matching between assets and liabilities, i.e., with $VA = VL$, the immunisation against interest rate risk is obtained if the instantaneous variance is equal to zero, that is, if:

$$\left\| \frac{1}{VA} \frac{\partial VA}{\partial \underline{X}'} \cdot \sigma(\underline{X}) \right\|^2 = \left\| \frac{1}{VL} \frac{\partial VL}{\partial \underline{X}'} \cdot \sigma(\underline{X}) \right\|^2 ,$$

where $\| \cdot \|$ represents the Euclidean norm in Re^n. Hence, hedging against interest rate risk involves matching the assets and liabilities sensitivities with respect to each model factor. The vectors $\frac{\partial VA}{\partial \underline{X}'} \frac{1}{VA}$ and $\frac{\partial VL}{\partial \underline{X}'} \frac{1}{VL}$ are usually known as the "stochastic durations" of assets and liabilities, respectively. Each element of such stochastic duration vectors is easily shown to correspond to a weighted average of the stochastic durations of its assets or liabilities components, with respect to a model state variable. Therefore, all that it is required is to know how to compute each stochastic duration for a

[17] The drift specification is a direct consequence of Q being a martingale measure.

discount factor. In the CIR model, $\dfrac{\partial P(r,\tau)}{\partial r} = -P(r,\tau)B(\tau)$. In the L&S model, $\dfrac{\partial P(r,v,\tau)}{\partial r} = P(r,v,\tau)C(\tau)$ and $\dfrac{\partial P(r,v,\tau)}{\partial v} = P(r,v,\tau)D(\tau)$.

In order to assign a temporal dimension to the above-specified stochastic durations, as well as to maintain consistency with the deterministic analysis made in section 3, the empirical analysis that will follow uses the stochastic duration definition of Munk (1999). That is, the stochastic duration of any asset or liability is rewritten as the time-to-maturity of a pure discount bond, with the same instantaneous variance of relative price changes. This measure is then directly compared with the Macaulay durations computed in section 3.

5 Data and Methodology

After the presentation of the theoretical background required for our study, the next stage was to estimate the parameters of the CIR and L&S models. As the benchmarks for Euroland interest rates, we used the O/N, 1-m, 3-m, 6-m, 1-yr Euribor rates and prices of German tradable debt on the following dates: 3 April and 7 November 1999. These dates were chosen because the European Central Bank changed its repo rate by 50bp, respectively, by a cut and a raise. In order to ensure the quality of the fitness, we considered only a 10-year investment horizon, which does not change the validity of the conclusions. All estimation routines were carried out through an adjustment to the dirty prices of the bonds and not in yield-to-maturity terms. This procedure obliged us to create synthetic assets for the Euribor rates. For both models, the parameters were estimated using non-linear least squares and no additional constraints were imposed, besides those strictly needed for the achievement of consistent theoretical solutions (and their implementation). The CIR model only requires that all parameters need to be positive and that $\alpha + \beta < 1$. However, the parameterisation of the L&S model is substantially more difficult. The model has 9 parameters, the two state variables r and v plus 7 parameters, $\alpha, \beta, \gamma, \eta, \delta, \lambda$ and ξ. Because of convergence difficulties, we followed the same procedures that Dahlquist and Svensson [1996] adopted to resolve this problem. Since the state variable associated with the short-term rate, r, can be associated to the overnight rate, the restriction that